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Weyl-ordered series form for the angle variable of the time-dependent oscillator

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Abstract
The definition of the phase variable for a classical time-dependent oscillator as the natural variable canonically conjugated to the Ermakov invariant is revised. Some implications of the result at the quantum level are discussed and an exact formal expression in terms of Weyl-ordered operators is given for the associated phase operator.

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1. Introduction

Since Dirac's early attempt [1], the problem of constructing well-behaved quantum phase operators has been approached by several authors and by resorting to very different physical and mathematical perspectives (see, e.g. [2, 4] and references therein). Several questions are raised, in fact, while tackling the problem of satisfactorily defining quantum phase operators, even in the simplest case represented by modes for the standard harmonic oscillator. For the harmonic oscillator (as well as for other systems described by time-independent Hamiltonians) one would expect the Hamiltonian to generate phase shifts, and hence a Hermitian phase operator conjugated to it. The impossibility of defining such an operator on $L^2(R)$ satisfying a canonical commutation relation with the Hamiltonian on a dense domain which includes the Hermite functions has been however recognized for the harmonic oscillator [2]. Basic problems are concerned with the periodicity of the phase and the existence of a lower bound to the energy. From a different perspective, difficulties can be seen to arise as the manifestation of a quantum anomaly problem: annihilation and creation operators of the harmonic oscillator satisfy the index relation $\dim\ker a^\dagger a - \dim\ker aa^\dagger = 1$, whereas a zero index would be required for a Hermitian phase operator (see discussion in [5]). In spite of all this, acceptable physical grounds have been invoked which enable one to effectively tackle the problem by defining likely criteria supporting pragmatic ways to proceed; for instance based on the use of the coherent state representation [6], on the introduction of approximate polar decomposition of the annihilation and creation operators and nonunitary phase operators [7], on the adoption of finite-dimensional Hilbert space [8], on the distinction among *ideal* and *feasible* phases [9] and so forth. In principle, criteria can be generalized to quantum systems more complicated than the harmonic oscillator, but to this aim it has been suggested to better exploit (classical and quantum) action–angle variables [10]. For non-autonomous systems, the motivation basically relies on the circumstance that resorting to quantum invariants had already been discussed and been found to be useful in constructing a solution to the Schrödinger equation and in analyzing the quantum dynamics features [11–13]. Clearly, basic conceptual problems still remain. Besides, further caution is required in that the nonlinear canonical transformation from position–momentum to action–angle coordinates are generally nonbijective (see, e.g. discussion in [14]). However, a careful adoption of the suggestion appears to be useful in that it would allow, at least, for a generalization of fundamental definition and effective tools which have been developed dealing with the standard harmonic oscillator (thus allowing, for instance, for the construction of a time-dependent operator of the Turski-type, of the Susskind–Glogower-type, etc). The idea has been followed by mainly focusing on the case of time-dependent oscillators since they have widespread applications in physics, entering into the description of very different systems such as atomic ensembles [15], gravitational waves [16, 17], particles in Paul traps [18], effective descriptions of unstable systems [19], etc. In [20], for instance, it has been remarked that once two invariants I_1, I_2 for a general time-dependent quadratic Hamiltonian are known a polar decomposition of the invariant $I = \sqrt{I_1^2 + I_2^2}$ can be considered whose phase variable $\tan^{-1} \frac{I_2}{I_1}$ obeys the canonical Poisson brackets $\{\phi, I\}_{p,q} = 1$, while in [21] the quantum Ermakov invariant for the parametric oscillator has been expressed in amplitude and phase variables and creation and annihilation operators have been written in the form $a = \sqrt{\hat{I}}\,\mathrm{e}^{-\mathrm{i}\hat{\Phi}}$, $a^\dagger = \mathrm{e}^{\mathrm{i}\hat{\Phi}}\sqrt{\hat{I}}$. Unfortunately, these studies apparently do not effectively clarify the picture in respect of the classical angle–action variables for the time-dependent oscillator from the phase space point of view. The main limitation in most of these studies is represented by the definition of the phase as a time-dependent function (the quantity $\theta(t)$ in equation (10) below), rather than a dynamical variable in phase space. But this step would give hints to a natural formulation of the quantum phase problem for the time-dependent oscillator, which would naturally allow us, for instance, to resort to quasiprobability distribution functions to provide quantum averages in a form which resembles classical averages. The main scope of this paper is to explore immediate consequences that would show up at the quantum level after defining in phase space the angle variable for the time-dependent oscillator as a dynamical variable in phase space.

2. Classical action–angle variables for the time-dependent oscillator

It is useful to recall first that, when considered as a dynamical variable in phase space, the phase of a classical one-dimensional standard harmonic oscillator, with constant mass m_0 and constant frequency ω_0, reads

$$\theta_0 = \tan^{-1}\left(\frac{m_0\omega_0 q}{p}\right), \tag{1}$$

where q and p denote the position coordinate and its conjugate momentum, respectively. Further, the classical Hamiltonian

$$H_0 = \frac{p^2}{2m_0} + \frac{m_0\omega_0^2 q^2}{2}, \tag{2}$$

can be used as the action variable canonically conjugate to the classical phase (1) up to a constant scaling. The transformation $(q, p) \to \left(\theta_0, \omega_0^{-1} H\right)$ is indeed canonical,

$\left\{\theta_0, \omega_0^{-1} H_0\right\}_{q,p} = 1$. The result holds no longer once a generalized oscillator having mass $m = m(t)$ and frequency $\omega = \omega(t)$ arbitrarily depending on time is considered. In such a case, the phase is obviously no longer given by equation (1) and the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \tag{3}$$

can no longer be an action variable because $\frac{\partial H}{\partial t} \neq 0$. Since the Hamiltonian is time-dependent and the naive time-invariance is lost, more complicated symmetry group and conservation law have to be considered, in fact. Angle–action variables for the time-dependent oscillator (3) can be easily obtained as a simple exercise, though. A route can be followed owing to what is known about the possibility to associate with a time-dependent oscillator a basic quadratic invariant. It is the so-called Ermakov invariant

$$I = \kappa \frac{y^2}{\sigma^2} + (\dot{\sigma} y - \dot{y}\sigma)^2, \qquad y = \sqrt{m} q, \tag{4}$$

where κ is an arbitrary positive constant and σ is a time-dependent function obeying the following equation of the Ermakov-type (see, e.g. [22, 23])

$$\ddot{\sigma} + \left[\omega^2 - \frac{\dot{M}}{2} - \frac{M^2}{4}\right]\sigma = \frac{\kappa}{\sigma^3}, \qquad M = \frac{\dot{m}}{m}. \tag{5}$$

The link of the Ermakov invariant I, which in terms of position and momentum variables takes the form

$$I = \kappa \frac{m}{\sigma^2} q^2 + \left[\frac{\sigma}{\sqrt{m}} p - \sqrt{m}\sigma q \frac{\mathrm{d}}{\mathrm{d}t}\left(\ln \frac{\sigma}{\sqrt{m}}\right)\right]^2, \tag{6}$$

with the symmetries underlying the time-dependent oscillator dynamics has been detailed in [23]. There it has been explicitly shown that the invariant I arises as the Noether invariant generated by a vector field of the type

$$V = \frac{\sigma^2}{\sqrt{\kappa}} \left\{\partial_t + \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln \frac{\sigma}{\sqrt{m}}\right)\right] q \partial_q\right\}. \tag{7}$$

For general non-autonomous systems, symmetries are associated with vector fields whose components along the time-derivative directions are not constant and depend on (q, p, t), in fact. The vector field V and the invariant I stand indeed for the time-dependent oscillator just like the time-derivative operator and H_0 stand for the standard harmonic oscillator. As a consequence, in the limit of constant mass and frequency the invariant I assumes a form, say I_0, that basically provides the Hamiltonian for the standard harmonic oscillator, $2\sqrt{\kappa} H_0 = \omega_0 I_0$. The quantity $J = \frac{I}{2\sqrt{\kappa}}$ can be therefore taken as the natural action variable for the time-dependent oscillator[1] and the associated angle variable θ is deduced by solving the partial differential equation $\{\theta, J\}_{q,p} = 1$. In doing so, it is easily obtained that in phase space the angle variable θ takes the form

$$\theta = \tan^{-1}\left[\frac{\sqrt{\kappa}}{\sigma^2}\left(\frac{p}{mq} - \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\sigma}{\sqrt{m}}\right)^{-1}\right]. \tag{8}$$

Expression (1) is recovered as a special case since the standard harmonic oscillator is concerned with $\sigma^2 = \frac{\sqrt{\kappa}}{\omega_0}$, (see equations (5)–(10)). Result (8) could have been achieved directly from

[1] The presence of the factor $2\sqrt{\kappa}$ supports for the choice $\kappa = 1/4$, which can be found for instance in [16, 17, 23–25], as the most economical.

what is known about the general solution for the dynamical differential equations associated with the Hamiltonian (3), i.e. $\frac{dq}{dt} = \frac{p}{m}$ and $\frac{dp}{dt} = -m\omega^2 q$, or equivalently

$$\frac{d^2 q}{dt^2} + \frac{d\ln m}{dt}\frac{dq}{dt} + \omega^2 q = 0. \tag{9}$$

It turns out, in fact, that the general solution to equation (9) can be put into the form of a mode having both amplitude ℓ and phase θ varying in time according to

$$q = \ell\cos(\theta+\delta) \qquad \text{with} \quad \ell = C\frac{\sigma}{\sqrt{m}} \qquad \theta = \sqrt{\kappa}\int^t \frac{dt'}{\sigma(t')^2}, \tag{10}$$

where δ and C are constants, and σ is the same time-dependent function as introduced before. Regardless knowledge of the explicit form for the solution to the differential equation (5) in specific cases (that is easily shown to be expressible as the square root of a proper bilinear combination of two independent solutions to the parametric oscillator equation obtained by setting its rhs equal to zero [26]), something can be said on some of the features which may be exhibited by dynamical variables when evaluated in phase space along solution trajectories for the time-dependent oscillator. As for the phase θ defined above, it can be easily rewritten as a dynamical variable in phase space thus obtaining (8), or equivalently

$$\theta = \tan^{-1}\left[\frac{C^2\sqrt{\kappa}}{\ell^2}\left(\frac{p}{q} - m\frac{d}{dt}\ln\ell\right)^{-1}\right], \tag{11}$$

so that $\{\theta, J\}_{q,p} = 1$. The transformation can be inverted and $\{q(\theta, J), p(\theta, J)\}_{\theta,J} = 1$. Hence, similarly to what happens for the Hamiltonian in the time-independent harmonic oscillator case, in the case of a time-dependent oscillator the Ermakov invariant basically provides the action variable $J = I/2\sqrt{\kappa}$ conjugate to the angle variable θ naturally introduced by equation (8) after inspection of the dynamics in phase space (i.e. of the behavior of the dynamical quantities q, p once the general structure for q has been recognized in the form of a mode having nontrivial time-dependent amplitude and phase). To our knowledge, the possibility to argue in the above fashion about the straight generalization of the oscillator phase definition (1) to form (8) is missing in the literature, even though a structural characterization of both solutions and symmetries the time-dependent oscillator differential equation (9) is well established. It is rather surprising that not even arguing on the general solution to the simple exercise $\{\tilde{\theta}(q, p, t), I(q, p, t)\}_{q,p} = \text{const}$ has been (apparently) considered so far in the literature. The closest formula we found has been written for a unit mass parametric oscillator (which is obviously equivalent to system (3) up to a simple linear canonical transformation) and has been derived by means of the proper time-dependent generating function in [27].

Before considering the definition of the phase at the quantum level, a comment is probably due since, at a first sight, one may not find fully satisfactory expression (8) and may be tempted to say that the argument of the derivative term is not completely independent from phase–space variables. At first glance one may even say that the experience accumulated so far with time-dependent oscillators, both at the classical and at the quantum level, has already demonstrated that the function σ can be considered as an auxiliary function inheriting informations on mass and frequency and playing a role that is basically expected for time-dependent frequency (see, e.g. [16, 21, 23, 24, 28, 29]). But the question can be better elucidated from the point of view of the needed time diffeomorphism, that is in terms of the symmetries underlying the dynamics of the time-dependent oscillator (3). Mapping the non-autonomous system (3) into an autonomous one implies, indeed, the search for a canonical transformation $(q, p, t) \to (Q(q, p, t), P(q, p, t), \tau(q, p, t))$ into the time-extended phase space (q, p, t) such that the vector field $\vec{V}$ turns into a derivative w.r.t. the time τ, $\vec{V} \rightarrow \partial_\tau$. It is a simple matter to infer that the last request is accomplished whenever one has

$$Q = Q\left(\sqrt{m}\frac{q}{\sigma}\right), \qquad \tau = \sqrt{\kappa}\int_{t_0}^{t}\frac{\mathrm{d}t'}{\sigma^2(t')} + g\left(\sqrt{m}\frac{q}{\sigma}\right) \tag{12}$$

(with g arbitrary function) and that demanding canonicity of $(q, p) \rightarrow (Q, P)$ yields to linearity of Q w.r.t. its argument and to the constant g. That is, the Ermakov invariant is just like the Hamiltonian $(Q^2 + P^2)/2$ for a unit mass and frequency harmonic oscillator whose position and momentum coordinates are

$$Q = \kappa^{1/4}\sqrt{m}\frac{q}{\sigma}, \qquad P = \frac{\sigma}{\kappa^{1/4}\sqrt{m}}p - q\frac{\sqrt{m}\sigma}{\kappa^{1/4}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\sigma}{\sqrt{m}}\right),$$

and whose actual time variable is $\tau = \sqrt{\kappa}\int^t \sigma^{-2}\,\mathrm{d}t'$. A different time scale is indeed generated through the time-diffeomorphism and formula (8) corresponds nothing but to $\tan^{-1} P/Q$.[2] Finally, recall that correlating a non-autonomous system to an autonomous one can be performed within the extended phase–space formalism (see, e.g. [30]). In such a case, the extended canonical transformation $(q, p, t, -H) \rightarrow (Q, P, \tau, -J)$ carrying the time-dependent quadratic Hamiltonian (3) into the harmonic oscillator form $J(Q, P) = (P^2 + Q^2)/2$ would read [31] as follows:

$$F_2(q, P, t, -J) = \kappa^{1/4}\sqrt{m}\frac{q}{\sigma}P + \frac{m}{2\kappa^{1/4}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}\right)q^2 - \sqrt{\kappa}J\int^t \sigma^{-2}\,\mathrm{d}t'. \tag{13}$$

However, inasmuch the formalism is assumed as the basis for the classical formulation of the dynamics, then in moving at the quantum level one would be thus faced with a quantum time operator acting on the same footing of the coordinates operators, and canonically conjugated to (minus) the Hamiltonian operator.

3. Features of the quantum phase and action operators for the time-dependent oscillator

The quantum phase problem for the standard harmonic oscillator is usually attacked either using number shifts operators, or using phase–space distributions and quantizing the classical phase, equation (1), by means of ordering rules for the momentum and position operators. It is worth recalling that in the latter case each Hermitian phase operator $\hat{\phi}$ such that the phase distribution

$$P(\varphi) = \mathrm{tr}[\delta(\hat{\phi} - \varphi)\hat{\rho}] \tag{14}$$

attributes the correct sharp phase to any large amplitude localized state $\hat{\rho}$ is expressible as the operator obtained from (1) by direct quantization of phase–space variables and introduction of an ordering rule (see, for instance, [32] and references therein). The relevance of equations (10) relies on the possibility to adopt a similar strategy in the case of the time-dependent oscillator as well. A likely way to tackle the problem of the definition of the quantum phase operator for the time-dependent oscillator by taking account of (8) and (10) is therefore based on the introduction of an operator of the type

$$\hat{\theta} = \left\{\tan^{-1}\left[\frac{\sqrt{\kappa}}{\sigma^2}\hat{q}\left(\frac{\hat{p}}{m} - \hat{q}\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}\right)^{-1}\right]\right\}_\Omega \tag{15}$$

where Ω means an operator ordering (e.g. the Weyl ordering). In this respect, a comment is in order. The appearance of $\frac{\mathrm{d}}{\mathrm{d}t}\ln\ell = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}$ in the classical angle variable (8), and thus

[2] Remark that the presence of an external driving force would be straightforwardly taken into account through slight algebraic changes.

in quantum phase operator (15), for a time-dependent oscillator sounds in fact to be highly meaningful from the physical point of view. At the quantum level the quantity measures in fact the departure from the minimum uncertainty of states that at an initial time are coherent but during their time evolution under the time-dependent oscillator dynamics generally become squeezed (as a consequence of a Bogolubov transformation mapping the Fock spaces at different times). It results indeed

$$\Delta_\alpha\hat{q}\Delta_\alpha\hat{p} \geqslant \frac{1}{2}\sqrt{1+\frac{\sigma^2}{\kappa}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}-\frac{\sigma}{2m}\frac{\mathrm{d}m}{\mathrm{d}t}\right)^2} = \frac{1}{2}\sqrt{1+\frac{\sigma^4}{\kappa}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}\right)^2}, \tag{16}$$

($\hbar = 1$) where the lower-script α is used to point out that the mean values are evaluated among eigenvectors of the time-dependent annihilation operator for the time-dependent oscillator obtained within the Lewis and Riesenfeld framework [23]. Minimum uncertainty is preserved during the time-evolution whenever the amplitude ℓ is constant, say $\sigma = c\sqrt{m}$, or equivalently when $m\omega = \frac{1}{2c^2}$ [25], c being a constant. In such a case, it would therefore result in

$$\hat{\theta}_{MU} = \tan^{-1}\left[\frac{\sqrt{\kappa}}{c^2}\frac{\hat{q}}{\hat{p}}\right]_\Omega, \qquad \hat{J}_{0,MU} = \frac{1}{2}\left(\frac{\sqrt{k}}{c^2}\hat{q}^2+\frac{c^2}{\sqrt{k}}\hat{p}^2\right) \tag{17}$$

while the Hamiltonian would read

$$\hat{H}_{MU} = \frac{1}{2m}\left(\hat{p}^2+\frac{\hat{q}^2}{4c^2}\right) \tag{18}$$

thus showing an explicit time-dependence. The action operator $\hat{J}_{0,MU}$ and the phase operator $\hat{\theta}_{MU}$ for these states would correspond to the Hamiltonian $\hat{H}_0$ and the phase operator $\hat{\theta}_0$ associated with the harmonic oscillator upon the identifications $\omega_0 = 1, m_0 = \sqrt{\kappa}/c^2$. Hence, even though they are described by non-stationary modes (their amplitude is constant but the angle argument of the mode does not vary linearly in time), minimum uncertainty phase states for the time-dependent oscillator can be considered as the closest analogs to the phase state for the quantum harmonic oscillator since in the two cases the natural quantum action–angle variables take the same form. Nevertheless, the Hamiltonian description of the two cases still require the adoption of two distinct time variables, the 'original' time t and the 'proper' time $\tau_{MU} = \frac{\sqrt{\kappa}}{c^2}\int^t[m(t')]^{-1}\,\mathrm{d}t'$.

4. Weyl-ordered polynomial quantum form for the angle variable

In this section we shall derive an explicit formal representation of the quantum angle operator for the time-dependent oscillator in terms of the position and momentum operators in analogy with the procedure outlined by Bender and Dunne in [33] for the harmonic oscillator, since the Hamilton equations for the time-dependent oscillator can be successfully integrated by exploiting the Ermakov invariant instead of the Hamiltonian. One has indeed

$$\dot{q} = \frac{p}{m} = \frac{1}{\sqrt{m}\sigma}\left[\sqrt{I-\kappa\frac{m}{\sigma^2}q^2}\right]+q\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}} \tag{19}$$

which after elementary manipulation yields to

$$\frac{\mathrm{d}z}{\sqrt{I-z^2}} = \sqrt{\kappa}\frac{\mathrm{d}t}{\sigma^2}, \qquad z = \sqrt{\kappa}\frac{m}{\sigma^2}q,$$

i.e. to $F(q,t) = t$ where

$$F(q,t) = \theta^{-1}\left[\int^{\sqrt{\kappa}\frac{m}{\sigma^2}q}\frac{\mathrm{d}z}{\sqrt{I-z^2}}\right] \tag{20}$$

(see equation (10)). Hence, one can wonder about solving a t-evolution equation of the Heisenberg-type for the quantum operators associated with the angle variable θ or its inverse F together with the invariance condition $\hat{I}_0(\hat{q}, \hat{p}, t) = \hat{I}_0(\hat{q}_0, \hat{p}_0, t_0)$, where q_0 is the value of q at the initial time t_0. On the other hand, one knows that the time-dependent oscillator actually becomes an autonomous system, and precisely the harmonic oscillator with unit mass and frequency, by introducing new position and momentum canonical variables and by redefining the time as previously discussed. Hence, one would get an angle operator $\hat{\theta}(\hat{Q}, \hat{P}, \tau)$ associated with the quantum form $\hat{J}(\hat{Q}, \hat{P})$ of the action variable according to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\theta}(\hat{Q}, \hat{P}) = -\mathrm{i}[\hat{\theta}(\hat{Q}, \hat{P}), \hat{J}(\hat{Q}, \hat{P})] = 1 \tag{21}$$

so that

$$\hat{\theta}(\hat{Q}, \hat{P}) = \left[\tan^{-1}\frac{\hat{Q}}{\hat{P}}\right]_{\Omega}. \tag{22}$$

As discussed in [33], to this phase operator a consistent formal meaning can be given in terms of a Weyl-ordered expansion with respect to the $\hat{Q}$ and $\hat{P}^{-1}$ operators. The Lewis–Riesenfeld approach might be followed and $\hat{q}$, $\hat{p}$ operators may be expressed as time-dependent linear functions of fixed-time position and momentum operators through a Bogolubov transformation [23]. But since $\hat{Q}$ and $\hat{P}$ depend linearly on both $\hat{Q}_0$ and $\hat{P}_0$ (or, equivalently, on both $\hat{a}(t_0)$ and $\hat{a}^\dagger(t_0)$), this would imply further Weyl expansions. In this respect, for a more practical purpose, the request can be formulated directly in the original phase–space operators $\hat{q}$, $\hat{p}$ as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\hat{\theta}}(\hat{q}, \hat{p}) = -\mathrm{i}[\hat{\hat{\theta}}(\hat{q}, \hat{p}), \hat{J}(\hat{q}, \hat{p})] = \hat{1} \tag{23}$$

to obtain the angle operator in terms of Weyl-ordered operators depending on $\hat{q}$ and $\hat{p}$. In doing so, we only need to bear in mind that the time variables in the two formulations are linked by a nontrivial diffeomorphism.

For the sake of convenience, we shall first formulate the problem of finding the angle function in the presence of a time diffeomorphism leading the original non-autonomous system into an autonomous one in a general form, namely for a generic time-dependent invariant quadratic in space–phase coordinates. So, after casting the invariant action operator in terms of the operators $\hat{q}$ and $\hat{p}$ as

$$\hat{J} = J(\hat{q}, \hat{p}, t) = J_{pp}(t)\hat{p}^2 + J_{qq}(t)\hat{q}^2 + J_{qp}(t)(\hat{p}\hat{q} + \hat{q}\hat{p}), \tag{24}$$

one leads to solve the differential operator equation

$$\Upsilon(t)\hat{1} = -\mathrm{i}[\hat{\theta}_W(\hat{q}, \hat{p}), \hat{J}(\hat{q}, \hat{p})], \qquad \Upsilon(t) = \frac{\mathrm{d}\tau}{\mathrm{d}t}, \tag{25}$$

for the angle function $\hat{\theta}_W(\hat{q}, \hat{p})$, the subscript being introduced to highlight that a Weyl-ordered form will be seeked. To this aim, it is useful to introduce a basis in terms of which the angle operator can be conveniently expressed (see also, discussion in [34]). It can be formed by starting from the ordinary Weyl recipe that replaces classical products $p^l x^n$ (with positive indices l and n) by the fully averaged and symmetrized operators $\hat{T}_{l,n}$ obtained by summing monomials of degree l in $\hat{p}$ and n in $\hat{q}$ according to[3]

$$\hat{T}_{l,n}(\hat{q}, \hat{p}) = \frac{1}{2^l}\sum_{s=0}^{\infty}\binom{l}{s}\hat{p}^s\hat{q}^n\hat{p}^{l-s} = \frac{1}{2^n}\sum_{s=0}^{\infty}\binom{n}{s}\hat{q}^s\hat{p}^l\hat{q}^{n-s}. \tag{26}$$

[3] Basis elements $\hat{T}_{i,j}$ can be equivalently written in other possibly convenient forms, e.g. [35]:

$$\hat{T}_{l,n} = \frac{(-i)^l}{2^l}\frac{\hat{N}!}{(\hat{N}-n)!}{}_2F_1(\hat{N}+1, -l; -1; \hat{N}-n+1)\hat{p}^{l-n}, \qquad \hat{N} = \mathrm{i}\hat{q}\hat{p}.$$

They satisfy

$$[\hat{T}_{l,n}, \hat{p}] = \mathrm{i}n\hat{T}_{l,n-1}, \qquad [\hat{T}_{l,n}, \hat{q}] = -\mathrm{i}l\hat{T}_{l-1,n}, \tag{27}$$

$$[\hat{T}_{l,n}, \hat{p}]_+ = 2\hat{T}_{l+1,n}, \qquad [\hat{T}_{l,n}, \hat{q}]_+ = 2\hat{T}_{l,n+1} \tag{28}$$

($[\cdot, \cdot]_+$ denotes the anticommutator). Obviously, classical dynamical variables may generally depend on negative powers of Hamiltonian coordinates (this is just the angle variable case, for instance), and so should their quantum analogs on the respective operators. The operator ordering prescription scheme can be straightforwardly extended to cover the cases for which formal quantum analogs of Laurent series are needed. One can indeed formally use the first-ordered form in equation (26) even for the case of positive momenta powers and negative coordinates powers, and the second-ordered form in equation (26) in the opposite case. Whenever both the power indices l, n are negatives, then binomials in (26) can be understood and expressed in terms of gamma functions, and the associated operator $\hat{T}_{l,n}$ takes an infinite series representation. After doing so, the commutation rules (27), (28) are still valid, and the set of all these operators $\hat{T}_{l,n}$ is assumed as a basis in the space of operators in $\hat{p}$ and $\hat{q}$ [33]. Once the angle operator $\hat{\theta}_W(\hat{q}, \hat{p})$ is demanded to be expressed as a time-dependent sum of all elements $\hat{T}_{l,n}(\hat{q}, \hat{p})$, say

$$\hat{\theta}_W(\hat{q}, \hat{p}) = \sum_{l,n\in\mathbb{Z}} \alpha_{l,n}(t) T_{l,n}(\hat{q}, \hat{p}), \tag{29}$$

then, owing to (27), (28), the time-dependent coefficients $\alpha_{l,n}(t)$ obey to

$$2\sum_{l,j\in\mathbb{Z}} \alpha_{l,n}[nJ_{pp}\hat{T}_{l+1,n-1} - lJ_{qq}\hat{T}_{l-1,n+1} + (n-l)J_{qp}\hat{T}_{ln}] = \Upsilon(t)\hat{T}_{00}. \tag{30}$$

This equation relates triplets of the $\alpha_{l,n}$'s and does not have a unique solution; the solution is determined up to a function of the action invariant, in fact. In the spirit of the study in [33], one can select the *minimal* solution, for which the set of nonvanishing $\alpha_{l,n}$'s is the smallest allowed under the coefficients difference constraints implied by equation (30). For the present case, the minimality condition requires

$$\alpha_{-1,1} = \frac{\Upsilon}{2J_{pp}}. \tag{31}$$

So, by using standard techniques, the minimal solution can be found of the form

$$\hat{\theta}_W(\hat{q}, \hat{p}) = \sum_{r=0}^{\infty} \alpha_{-r,r}(t)\, \hat{T}_{-r,r}(\hat{q}, \hat{p}) \tag{32}$$

where

$$\alpha_{0,0} = \frac{\Upsilon(t)}{2\sqrt{J_{pp}J_{qq} - J_{qp}^2}} \tan^{-1}\left[\frac{J_{qp}}{\sqrt{J_{pp}J_{qq} - J_{qp}^2}}\right], \tag{33}$$

$$\alpha_{-1-2r,1+2r} = \frac{\Upsilon(t)}{2} \frac{(J_{qp}^2 - J_{pp}J_{qq})^r}{(1+2r)J_{pp}^{1+2r}}\, {}_2F_1\left(-\frac{1}{2} - r, -r, \frac{1}{2}, \frac{J_{qp}^2}{J_{qp}^2 - J_{pp}J_{qq}}\right) \tag{34}$$

$$\alpha_{-2-2r,2+2r} = -\frac{\Upsilon(t)}{2} \frac{J_{qp}(J_{qp}^2 - J_{pp}J_{qq})^r}{J_{pp}^{2+2r}}\, {}_2F_1\left(-\frac{1}{2} - r, -r, \frac{3}{2}, \frac{J_{qp}^2}{J_{qp}^2 - J_{pp}J_{qq}}\right) \tag{35}$$

$(r = 0, 1, 2, \ldots)$. Applications of above formulae to systems associated with the Hamiltonian (3) proceeds through the identification $\hat{J} = \hat{I}/(2\sqrt{\kappa})$, i.e.

$$J_{pp} = \frac{\sigma^2}{2m\sqrt{\kappa}}, \qquad J_{qq} = \frac{m}{2}\left[\frac{\sigma^2}{\sqrt{\kappa}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}\right)^2 + \frac{\sqrt{\kappa}}{\sigma^2}\right],$$
$$J_{qp} = -\frac{\sigma^2}{2\sqrt{\kappa}}\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}, \qquad \Upsilon = \frac{\sqrt{\kappa}}{\sigma^2}. \tag{36}$$

Hence, for the general time-dependent oscillator we finally get

$$\alpha_{0,0} = \frac{\sqrt{\kappa}}{\sigma^2}\tan^{-1}\left[-\frac{\sigma^2}{\sqrt{\kappa}}\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}\right], \tag{37}$$

$$\alpha_{-1-2r,1+2r} = \frac{(-1)^r}{(1+2r)m}\left(\frac{\sqrt{\kappa}m}{\sigma^2}\right)^{2+2r} {}_2F_1\left[-\frac{1}{2}-r, -r, \frac{1}{2}, -\frac{\sigma^4}{\kappa}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}\right)^2\right] \tag{38}$$

$$\alpha_{-2-2r,2+2r} = (-1)^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}\right)\left(\frac{\sqrt{\kappa}m}{\sigma^2}\right)^{2+2r} {}_2F_1\left[-\frac{1}{2}-r, -r, \frac{3}{2}, -\frac{\sigma^4}{\kappa}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\sigma}{\sqrt{m}}\right)^2\right] \tag{39}$$

$(r = 0, 1, 2, \ldots)$. The way the coefficients depend on squeezing effects is manifest (see equations (16)). Remark also that the above series contains both even and odd 'powers' of $T_{-1,1} = \hat{q}\hat{p}^{-1} + \hat{p}^{-1}\hat{q}$, whereas the standard harmonic oscillator case is concerned with the latter only. The harmonic oscillator case is associated with a vanishing I_{qp} term, indeed. The solution found in [33] for the phase operator associated with the time-independent harmonic oscillator with unit mass and frequency, for which all the coefficients but the $\alpha_{-1-2r,1+2r} = (-1)^r(1+2r)^{-1}$ (with $r = 0, 1, 2\ldots$) are equal to zero, is easily recovered after insertion of $m = 1$ and $\sigma^2 = \sqrt{\kappa}$ in equations (37)–(39).

5. Conclusions

We reviewed the definition of the phase of a time-dependent oscillator as the variable canonically conjugate to the Ermakov action invariant. By virtue of this, we have been lead to remark that, just like expression (1) for the standard harmonic oscillator, the expression given by (8) may be employed as a basis to move toward the quantization of the phase of time-dependent oscillators. By taking into account that the mapping from the non-autonomous time-dependent Hamiltonian to the Ermakov action variable J actually implies a time diffeomorphism, we have consequently derived a Weyl-ordered expansion for the angle operator conjugated to the Ermakov action operator in terms of position and inverse momentum operators defined at a given initial time. In principle, this enables one to express the solution to the spectral problem at any given time in terms of the solution at the initial time, since the Bogolubov transformation between the two corresponding Fock spaces is known [23]. Efforts seem therefore due to investigate phase states associated with the phase operator (15) and to analyze how they are affected by the mechanism of loss of coherence, incidentally through a proper introduction of terms that take care of multivaluedness of the phase variable and by properly adapting, for instance, arguments dealt with in detail by Royer [32]. Obviously, much care is needed while concretely handling the angle operators due to the presence of a formal inverse momentum operator $\hat{p}^{-1}$ (and its powers). The interesting recent discussions in [36–40], where the problems of constructing a well-defined self-adjoint version and a positive operator-valued measure of the Aharonov–Bohm time operator $\hat{T}_{-1,1} = \frac{1}{2}(\hat{q}\hat{p}^{-1} + \hat{p}^{-1}\hat{q})$ have been studied, are very relevant in this respect. The extensions of the results of [36–40] to the operators $\hat{T}_{-2,2}, \hat{T}_{-3,3}, \ldots,$ as well as their applications to specific systems of the physical interest, are currently under investigation.

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